

Sekiguchi-Debiard operators at infinity

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Abstract: We construct a family of pairwise commuting operators such that the Jack symmetric functions of infinitely many variables x_1, x_2, \dots are their eigenfunctions. These operators are defined as limits at $N \rightarrow \infty$ of renormalised Sekiguchi-Debiard operators acting on symmetric polynomials in the variables x_1, \dots, x_N . They are differential operators in terms of the power sum variables $p_n = x_1^n + x_2^n + \dots$ and we compute their symbols by using the Jack reproducing kernel. Our result yields a hierarchy of commuting Hamiltonians for the quantum Calogero-Sutherland model with infinite number of bosonic particles in terms of the collective variables of the model. Our result also yields explicit shift operators for the Jack symmetric functions.

1. Introduction

1.1. Calogero-Sutherland model. This quantum model describes a system of N bosonic particles on a circle $\mathbb{R}/\pi\mathbb{Z}$ with the Hamiltonian [5, 19, 20]

$$H_{\text{CS}} = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial q_i^2} + \sum_{i < j} \frac{\beta(\beta-1)}{\sin^2(q_i - q_j)}$$

where $0 \leq q_1, \dots, q_N < \pi$. Being translationally invariant H_{CS} commutes with the momentum operator

$$P_{\text{CS}} = -i \sum_j \frac{\partial}{\partial q_j}.$$

After eliminating the vacuum factor

$$\omega = \left| \prod_{i < j} \sin(q_i - q_j) \right|^\beta$$

and then passing to the exponential variables $x_j = \exp(2iq_j)$ and the parameter $\alpha = \beta^{-1}$ more common in the mathematical literature, the Hamiltonian becomes

$$\omega^{-1} \circ H_{\text{CS}} \circ \omega = \frac{2}{\alpha} H_N^{(2)} + \frac{N^3 - N}{6\alpha^2}$$

where

$$H_N^{(2)} = \alpha \sum_i \left(x_i \frac{\partial}{\partial x_i} \right)^2 + \sum_{i < j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right). \quad (1.1)$$

Respectively, P_{CS} gives rise to the operator

$$H_N^{(1)} = \frac{1}{2} P_{\text{CS}} = \sum_j x_j \frac{\partial}{\partial x_j}.$$

The operators $H_N^{(1)}$ and $H_N^{(2)}$ commute and act on symmetric polynomials of the variables x_1, \dots, x_N . It is known that both operators can be included into a quantum integrable hierarchy, that is into a polynomial ring of commuting differential operators with N generators of orders $1, \dots, N$ respectively, called the Sekiguchi-Debiard operators [6, 8, 16]. The Jack symmetric polynomials [8] are joint eigenfunctions of the hierarchy. They are labelled by partitions of $0, 1, 2, \dots$ with no more than N non-zero parts.

In combinatorics it is quite common to extend various symmetric polynomials to an infinite countable set of variables. These extensions are called *symmetric functions*. In particular, the extensions of the Jack symmetric polynomials are well studied [8]. They are labelled by partitions of $0, 1, 2, \dots$. However, no explicit expressions for higher commuting Hamiltonians corresponding to the infinite set of variables have been yet available in the Jack case, with an exception of a few lower order operators. The main purpose of the present article is to fill up the gap, by studying the limits of the Sekiguchi-Debiard operators as $N \rightarrow \infty$ and giving explicit expressions for the resulting commuting Hamiltonians.

As another application of our result, we construct *elementary shift operators* for the Jack symmetric functions. In terms of the labels, our operators correspond to decreasing any given non-zero part of a partition by 1 and to the operation on partitions inverse to that, see our formulas (2.27) and (2.25) respectively. For the origins of this construction see the work [18] and references therein.

1.2. Collective variables. The standard way to treat the $N = \infty$ case is to rewrite the Hamiltonian (1.1) in terms of the power sums (the “collective variables” in the condensed matter physics terminology)

$$x_1^n + \dots + x_N^n \quad \text{where} \quad n = 1, \dots, N \quad (1.2)$$

and to take the limit at $N \rightarrow \infty$ afterwards. Denoting the limit of the power sum (1.2) by p_n where $n = 1, 2, \dots$ the resulting Hamiltonian reads [9]

$$\begin{aligned} H^{(2)} = \lim_{N \rightarrow \infty} \alpha (H_N^{(2)} - N H_N^{(1)}) = \\ \sum_{m,n=1}^{\infty} \left(\alpha (m+n) p_m p_n \frac{\partial}{\partial p_{m+n}} + \alpha^2 m n p_{m+n} \frac{\partial^2}{\partial p_m \partial p_n} \right) + \\ (\alpha - 1) \sum_{n=1}^{\infty} \alpha n^2 p_n \frac{\partial}{\partial p_n} . \end{aligned} \quad (1.3)$$

The first and the second summands in the middle line of the above display are known as *splitting terms* and *joining terms* respectively.

Consider the vector space $\Lambda = \mathbb{C}[p_1, p_2, \dots]$ and equip it with the operators a_n where $n \in \mathbb{Z} \setminus \{0\}$, defined on the polynomials $f \in \Lambda$ by

$$a_n(f) = \begin{cases} p_{-n} f & \text{if } n < 0, \\ \alpha n \partial f / \partial p_n & \text{if } n > 0; \end{cases}$$

see also the equality (2.14) below. The operators a_n satisfy the relations

$$[a_m, a_n] = m \alpha \delta_{m+n, 0} . \quad (1.4)$$

Thus Λ becomes a highest weight module for an infinite-dimensional Heisenberg Lie algebra. In terms of the operators a_n the Hamiltonian (1.3) takes the form

$$H^{(2)} = \sum_{m,n=1}^{\infty} (a_{-m} a_{-n} a_{m+n} + a_{-m-n} a_m a_n) + (\alpha - 1) \sum_{n=1}^{\infty} n a_{-n} a_n .$$

Similarly, the $H_N^{(1)}$ yields a first order differential operator commuting with $H^{(2)}$

$$H^{(1)} = \lim_{N \rightarrow \infty} \alpha H_N^{(1)} = \sum_{n=1}^{\infty} \alpha n p_n \frac{\partial}{\partial p_n} = \sum_{n=1}^{\infty} a_{-n} a_n . \quad (1.5)$$

1.3. *Quantum field theory.* In terms of the field $\varphi(s)$ on the circle $S^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$

$$\varphi(s) = \sum_{n \neq 0} a_n \exp(-i n s) \quad \text{for } s \in S^1$$

the commutation relations (1.4) read

$$[\varphi(s), \varphi(t)] = 2\pi i \alpha \delta'(s - t).$$

Setting $a_0 = 0$ and using the Wick normal ordering $:$ with the operators a_n for $n < 0$ to the left and for $n > 0$ to the right, the Hamiltonian $H^{(1)}$ becomes

$$\frac{1}{2\alpha} \sum_{m+n=0} : a_m a_n : = \frac{1}{2\alpha} \int_{S^1} \frac{ds}{2\pi} : \varphi^2(s) :$$

while $H^{(2)}$ becomes the Hamiltonian of quantized Benjamin-Ono equation [1, 13]

$$\begin{aligned} & \frac{1}{3} \sum_{l+m+n=0} : a_l a_m a_n : + \frac{\alpha - 1}{2} \sum_{m+n=0} |n| : a_m a_n : = \\ & \int_{S^1} \frac{ds}{2\pi} \left(\frac{1}{3} : \varphi^3(s) : + \frac{1 - \alpha}{2} : \varphi'(s) (\mathcal{H}\varphi)(s) : \right) \end{aligned}$$

where \mathcal{H} stands for the Hilbert transform

$$(\mathcal{H}\varphi)(s) = \text{p.v.} \int_{S^1} \frac{dt}{2\pi} \cot \frac{s - t}{2} \varphi(t).$$

In the particular case $\alpha = 1$ the Jack symmetric polynomials degenerate into Schur polynomials. The Benjamin-Ono equation respectively degenerates into the dispersionless KdV (also called Burgers) equation. An explicit construction of a countable set of commuting Hamiltonians for the quantum dispersionless KdV can be obtained via boson-fermion correspondence and is available in terms of recurrence relations [12] or a generating function [14].

The higher quantum Hamiltonians for any parameter α are constructed in the present article, see the theorem in Subsection 2.5. Note that in the case $\alpha = 1$ our Hamiltonians $A^{(k)}$ are different from those obtained in [12, 14] being rather their polynomial combinations, see for example the relations (2.16) and (2.17).

In their turn, Jack symmetric polynomials can be regarded as degenerations of the Macdonald polynomials in the variables x_1, \dots, x_N also depending on two formal parameters q and t . The Jack case corresponds to $q = t^\alpha$ where $t \rightarrow 1$. The Sekiguchi-Debiard operators can be then regarded as degenerations of the Macdonald operators [8] acting on the symmetric polynomials in x_1, \dots, x_N . Our theorem generalizes to the Macdonald case [10], see also the earlier works [2, 17].

1.4. Plan of the article. In the next section we recall some basic facts from the theory of symmetric functions and set up the notation. We then introduce the Jack polynomials and Sekiguchi-Debiard differential operators. Our main tool is the notion of the symbol of an operator relative to the reproducing kernel for Jack polynomials. After establishing the basics, we state our main result which is an explicit formula for the symbol of the generating function of commuting Hamiltonians. Then we explicitly construct our elementary shift operators for the Jack symmetric functions. We finish Section 2 with reducing the proof of our theorem to certain determinantal identities which are then proved in Section 3.

In this article we generally keep to the notation of the book [8] for symmetric functions. When using the results from [8] we will simply indicate their numbers within the book. For example, the statement (1.11) from Chapter I of the book will be referred to as [I.1.11] assuming it is from [8]. We do not number our own lemmas, propositions, theorems or corollaries because we have only one of each.

2. Symmetric functions

2.1. Monomial functions. Fix any field \mathbb{F} . For any positive integer N denote by Λ_N the \mathbb{F} -algebra of symmetric polynomials in N variables x_1, \dots, x_N . The algebra Λ_N is graded by the polynomial degree. The substitution $x_N = 0$ defines a homomorphism $\Lambda_N \rightarrow \Lambda_{N-1}$ preserving the degree. Here $\Lambda_0 = \mathbb{F}$. The inverse limit of the sequence

$$\Lambda_1 \leftarrow \Lambda_2 \leftarrow \dots$$

in the category of graded algebras is denoted by Λ . The elements of Λ are called *symmetric functions*. Following [8] we will introduce some standard bases of Λ .

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be any partition of $0, 1, 2, \dots$. The number of non-zero parts is called the *length* of λ and is denoted by $\ell(\lambda)$. If $\ell(\lambda) \leq N$ then the sum of all distinct monomials obtained by permuting the N variables in $x_1^{\lambda_1} \dots x_N^{\lambda_N}$ is denoted by $m_\lambda(x_1, \dots, x_N)$. The symmetric polynomials $m_\lambda(x_1, \dots, x_N)$ with $\ell(\lambda) \leq N$ form a basis of the vector space Λ_N . By definition, for $\ell(\lambda) \leq N$

$$m_\lambda(x_1, \dots, x_N) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \sum_{\sigma \in \mathfrak{S}_k} d_\lambda^{-1} x_{i_{\sigma(1)}}^{\lambda_1} \dots x_{i_{\sigma(k)}}^{\lambda_k} \quad (2.1)$$

where we write k instead of $\ell(\lambda)$. Here \mathfrak{S}_k is the symmetric group permuting the numbers $1, \dots, k$ and

$$d_\lambda = k_1! k_2! \dots \quad (2.2)$$

if k_1, k_2, \dots are the respective multiplicities of the parts $1, 2, \dots$ of λ . Further,

$$m_\lambda(x_1, \dots, x_{N-1}, 0) = \begin{cases} m_\lambda(x_1, \dots, x_{N-1}) & \text{if } \ell(\lambda) < N; \\ 0 & \text{if } \ell(\lambda) = N. \end{cases} \quad (2.3)$$

Hence for any fixed partition λ the sequence of polynomials $m_\lambda(x_1, \dots, x_N)$ with $N \geq \ell(\lambda)$ has a limit in Λ . This limit is called the *monomial symmetric function* corresponding to λ . Simply omitting the variables, we will denote the limit by m_λ . With λ ranging over all partitions of $0, 1, 2, \dots$ the symmetric functions m_λ form a basis of the vector space Λ . Note that if $\ell(\lambda) = 0$ then we set $m_\lambda = 1$.

2.2. Power sums. For each $n = 1, 2, \dots$ denote $p_n(x_1, \dots, x_N) = x_1^n + \dots + x_N^n$. When the index n is fixed the sequence of symmetric polynomials $p_n(x_1, \dots, x_N)$ with $N = 1, 2, \dots$ has a limit in Λ , called the *power sum symmetric function* of degree n . We denote the limit by p_n . More generally, for any partition λ put

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \quad (2.4)$$

where we set $p_0 = 1$. The elements p_λ form another basis of Λ . In other words, the elements p_1, p_2, \dots are free generators of the commutative algebra Λ over \mathbb{F} .

The basis of p_λ can be related to the basis of monomial symmetric functions as follows. For any two partitions λ and μ denote by $R_{\lambda\mu}$ the number of mappings $\theta : \{1, \dots, \ell(\mu)\} \rightarrow \{1, 2, \dots\}$ such that

$$\sum_{\theta(j)=i} \mu_j = \lambda_i \quad \text{for each } i = 1, 2, \dots \quad (2.5)$$

For any such θ the partition μ in (2.5) is called a *refinement* of λ . Note that if $R_{\lambda\mu} \neq 0$ then λ and μ are partitions of the same number. Moreover, then by [I.6.10] we have $\mu \leq \lambda$ in the *natural partial ordering* of partitions:

$$\mu_1 \leq \lambda_1, \quad \mu_1 + \mu_2 \leq \lambda_1 + \lambda_2, \quad \dots$$

By [I.6.9] we have

$$p_\mu = \sum_{\lambda} R_{\lambda\mu} m_\lambda. \quad (2.6)$$

2.3. Jack functions. Now let \mathbb{F} be the field $\mathbb{Q}(\alpha)$ where α is another variable. Define a bilinear form $\langle \cdot, \cdot \rangle$ on the vector space Λ by setting for any λ and μ

$$\langle p_\lambda, p_\mu \rangle = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda\mu} \quad (2.7)$$

where

$$z_\lambda = 1^{k_1} k_1! 2^{k_2} k_2! \dots$$

in the notation (2.2). This form is obviously symmetric and non-degenerate. By [Ex. VI.4.2] there exists a unique family of elements $P_\lambda \in \Lambda$ such that

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{for } \lambda \neq \mu$$

and such that any P_λ equals m_λ plus a linear combination of the elements m_μ with $\mu < \lambda$ in the natural partial ordering. The elements $P_\lambda \in \Lambda$ are called the *Jack symmetric functions*. Alternatively, they can be defined as follows.

Denote by $\Delta(x_1, \dots, x_N)$ the *Vandermonde polynomial* of N variables

$$\det \left[x_i^{N-j} \right]_{i,j=1}^N = \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

Put

$$S_N(u) = \Delta(x_1, \dots, x_N)^{-1} \cdot \det \left[x_i^{N-j} (u + j - 1 - \alpha x_i \partial_i) \right]_{i,j=1}^N \quad (2.8)$$

where u is a variable and ∂_i is the operator of partial derivation relative to x_i . Here the determinant is defined as the alternated sum

$$\sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma \prod_{i=1}^N (x_i^{N-\sigma(i)} (u + \sigma(i) - 1 - \alpha x_i \partial_i)) \quad (2.9)$$

where as usual $(-1)^\sigma$ denotes the sign of permutation σ . In every product over $i = 1, \dots, N$ appearing in (2.9) the operator factors pairwise commute, hence their ordering does not matter. Further, $S_N(u)$ is a polynomial in the variable u with pairwise commuting operator coefficients preserving the space Λ_N , see for instance [Ex. VI.3.1]. We will call the restrictions of these coefficients to the space Λ_N the *Sekiguchi-Debiard operators*. By [Ex. VI.4.2] the latter operators have a common eigenbasis in Λ_N parametrized by partitions λ of length $\ell(\lambda) \leq N$. The eigenvectors are called the *Jack symmetric polynomials*.

For each λ with $\ell(\lambda) \leq N$ there is an eigenvector denoted by $P_\lambda(x_1, \dots, x_N)$ which is equal to $m_\lambda(x_1, \dots, x_N)$ plus a linear combination of the polynomials $m_\mu(x_1, \dots, x_N)$ with $\mu < \lambda$ and $\ell(\mu) \leq N$. It turns out that each coefficient in this linear combination does not depend on N . Note that if λ and μ are any two partitions of the same number such that $\lambda \geq \mu$, then $\ell(\lambda) \leq \ell(\mu)$ due to [I.1.11]. It follows that the polynomials $P_\lambda(x_1, \dots, x_N)$ enjoy the same *stability property* as the polynomials $m_\lambda(x_1, \dots, x_N)$ in (2.3):

$$P_\lambda(x_1, \dots, x_{N-1}, 0) = \begin{cases} P_\lambda(x_1, \dots, x_{N-1}) & \text{if } \ell(\lambda) < N; \\ 0 & \text{if } \ell(\lambda) = N. \end{cases} \quad (2.10)$$

In particular, the sequence of polynomials $P_\lambda(x_1, \dots, x_N)$ with $N \geq \ell(\lambda)$ has a limit in Λ . This is exactly the Jack symmetric function P_λ . The eigenvalues of Sekiguchi-Debiard operators acting on Λ_N are also known. By [Ex. VI.4.2]

$$S_N(u) P_\lambda(x_1, \dots, x_N) = \prod_{i=1}^N (u + i - 1 - \alpha \lambda_i) \cdot P_\lambda(x_1, \dots, x_N). \quad (2.11)$$

2.4. Reproducing kernel. In this subsection we will regard the elements of Λ as infinite sums of finite products of the variables x_1, x_2, \dots . For instance, we have

$$p_n = x_1^n + x_2^n + \dots$$

for any $n \geq 1$. When we need to distinguish x_1, x_2, \dots from any other variables, we will write $f(x_1, x_2, \dots)$ instead of any $f \in \Lambda$. Now let y_1, y_2, \dots be variables independent of x_1, x_2, \dots . According to [VI.10.4] with the bilinear form (2.7) one associates the *reproducing kernel*

$$\Pi = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1/\alpha}. \quad (2.12)$$

This Π should be regarded as an infinite sum of monomials in x_1, x_2, \dots and in y_1, y_2, \dots by expanding the factor corresponding to i, j as a series at $x_i y_j \rightarrow 0$.

The property of Π most useful for us can be stated as the following lemma. For any $f \in \Lambda$ denote by f^* the operator on Λ adjoint to the multiplication by f relative to the bilinear form (2.7). Note that here $f = f(x_1, x_2, \dots)$.

Lemma. *We have*

$$f^*(\Pi)/\Pi = f(y_1, y_2, \dots). \quad (2.13)$$

Proof. The commutative algebra Λ is generated by the elements p_n with $n \geq 1$. Therefore it suffices to prove (2.13) for $f = p_n$ only. Consider the operator $\partial/\partial p_n$ of derivation in Λ relative to $p_n = p_n(x_1, x_2, \dots)$. By the definition (2.7) we have

$$p_n^* = \alpha n \partial/\partial p_n \quad (2.14)$$

On the other hand, by taking the logarithm of (2.12) and then exponentiating,

$$\Pi = \exp \left(\sum_{n=1}^{\infty} p_n(x_1, x_2, \dots) p_n(y_1, y_2, \dots) / \alpha n \right).$$

The relation (2.13) for $f = p_n$ follows from the last two displayed equalities. \square

2.5. Main result. Let $\mathbb{F} = \mathbb{Q}(\alpha)$ as in the previous two subsections. For $N \geq 1$ let ρ_N be the homomorphism $\Lambda_N \rightarrow \Lambda_{N-1}$ defined by setting $x_N = 0$, as in the beginning of Subsection 2.1. Denote

$$A_N(u) = S_N(u)/(u)_N \quad (2.15)$$

where we employ the *Pochhammer symbol*

$$(u)_N = u(u+1)\dots(u+N-1).$$

The right hand side of the equation (2.15) is regarded as a rational function of u with the values being operators acting on the space Λ_N . Due to the stability property (2.10) of Jack symmetric polynomials, the equation (2.11) implies that

$$\rho_N A_N(u) = A_{N-1}(u) \rho_N$$

where $A_0(u) = 1$. So the sequence of $A_N(u)$ with $N \geq 1$ has a limit at $N \rightarrow \infty$. This limit can be written as a series

$$A(u) = 1 + A^{(1)}/(u)_1 + A^{(2)}/(u)_2 + \dots$$

where the coefficients $A^{(1)}, A^{(2)}, \dots$ are certain linear operators acting on Λ . By definition, the Jack symmetric functions are joint eigenvectors of these operators. In particular, the operators $A^{(1)}, A^{(2)}, \dots$ pairwise commute, and are self-adjoint relative to the bilinear form (2.7). We call them the *Sekiguchi-Debiard operators at infinity*. Due to the property (2.10) their definition immediately implies that

$$A^{(k)} P_\lambda = 0 \quad \text{if } \ell(\lambda) < k.$$

It is also transparent from (2.11) that for any homogeneous $f \in \Lambda$

$$A^{(1)} f = -\alpha \deg f.$$

Hence in the notation (1.5)

$$-A^{(1)} = H^{(1)}. \quad (2.16)$$

The operator $A^{(2)}$ is well studied [Ex. VI.4.3]. In particular, it is known that

$$A^{(1)}(A^{(1)} + 1) - 2A^{(2)} = H^{(2)} \quad (2.17)$$

in the notation (1.3). The main result of our article is the more general

Theorem. *In the notation (2.2) for each $k = 1, 2, \dots$ we have*

$$A^{(k)} = (-1)^k \sum_{\ell(\lambda)=k} d_\lambda m_\lambda m_\lambda^* \quad (2.18)$$

where λ ranges over all partitions of length k .

By inverting the relation (2.6) any monomial symmetric function m_λ can be expressed as a linear combination of the functions p_μ where λ, μ are partitions of the same number and $\lambda \leq \mu$. By substituting into (2.18) and using (2.4), (2.14) one can write each operator $A^{(k)}$ in terms of p_n and $\partial/\partial p_n$ where $n = 1, 2, \dots$. In particular, one recovers the above formulas for the operators $A^{(1)}$ and $A^{(2)}$.

2.6. Shift operators. In this subsection we will get a corollary to our theorem by using the following particular case of the *Pieri rule* for Jack symmetric functions. By [VI.6.24] for any partition μ the product $p_1 P_\mu$ equals the linear combination of the symmetric functions P_λ with the coefficients

$$\prod_{j=1}^{i-1} \frac{\alpha(\lambda_i - \lambda_j) - i + j - 1}{\alpha(\lambda_i - \lambda_j - 1) - i + j} \cdot \prod_{j=1}^{i-1} \frac{\alpha(\lambda_i - \lambda_j - 1) - i + j + 1}{\alpha(\lambda_i - \lambda_j) - i + j} \quad (2.19)$$

where λ ranges over all partitions such that the sequence $\lambda_1, \lambda_2, \dots$ is obtained from μ_1, μ_2, \dots by increasing one of its terms by 1 and i is the index of the term.

Further, by [VI.6.19] the above stated equality implies that for any partition λ the symmetric function $\partial P_\lambda / \partial p_1 = \alpha^{-1} p_1^* P_\lambda$ equals the linear combination of the P_μ with the coefficients

$$\prod_{j=1}^{\lambda_i-1} \frac{\alpha(\lambda_i - j - 1) + \lambda'_j - i + 1}{\alpha(\lambda_i - j) + \lambda'_j - i} \cdot \prod_{j=1}^{\lambda_i-1} \frac{\alpha(\lambda_i - j + 1) + \lambda'_j - i}{\alpha(\lambda_i - j) + \lambda'_j - i + 1} \quad (2.20)$$

where μ ranges over all partitions such that the sequence μ_1, μ_2, \dots is obtained from $\lambda_1, \lambda_2, \dots$ by decreasing one of its terms by 1 and i is the index of the term. As usual, here $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is the partition conjugate to λ .

Now define

$$B(u) = B^{(1)}/(u)_1 + B^{(2)}/(u)_2 + \dots$$

where $B^{(1)}, B^{(2)}, \dots$ are the linear operators acting on Λ determined by setting

$$B^{(k+1)} = (-1)^k \sum_{\ell(\lambda)=k} d_{\lambda \sqcup 1} m_{\lambda \sqcup 1} m_\lambda^*$$

where $\lambda \sqcup 1$ denotes the partition obtained from λ by appending one extra part 1. Also define

$$C(u) = C^{(1)}/(u)_1 + C^{(2)}/(u)_2 + \dots$$

where

$$C^{(k+1)} = (-1)^k \alpha^{-1} \sum_{\ell(\lambda)=k} d_{\lambda \sqcup 1} m_\lambda m_{\lambda \sqcup 1}^*.$$

Note that $B(u)^* = \alpha C(u)$. Moreover, we will have a corollary to our theorem.

By the definition (2.15) of the series $A(u)$, for any partition λ we have

$$A(u) P_\lambda = \prod_{i=1}^{\infty} \frac{u + i - 1 - \alpha \lambda_i}{u + i - 1} \cdot P_\lambda, \quad (2.21)$$

see (2.11). In the infinite product displayed above the only factors different from 1 are those corresponding to the indices $i = 1, \dots, \ell(\lambda)$. For any such i consider the product

$$\frac{1}{u + i - 1} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{u + j - 1 - \alpha \lambda_j}{u + j - 1}. \quad (2.22)$$

Corollary. (i) *For any given partition μ we have the equality*

$$B(u) P_\mu = \sum_{\lambda} B_{\lambda\mu}(u) P_\lambda$$

where $B_{\lambda\mu}(u)$ is the product of (2.19) by (2.22) while λ ranges over all partitions such that the sequence $\lambda_1, \lambda_2, \dots$ is obtained from μ_1, μ_2, \dots by increasing one of its terms by 1 and i is the index of the term.

(ii) *For any given partition λ we have the equality*

$$C(u) P_\lambda = \sum_{\mu} C_{\mu\lambda}(u) P_\mu$$

where $C_{\mu\lambda}(u)$ is the product of (2.20) by (2.22) while μ ranges over all partitions such that the sequence μ_1, μ_2, \dots is obtained from $\lambda_1, \lambda_2, \dots$ by decreasing one of its terms by 1 and i is the index of the term.

Proof. By [Ex. I.5.3] our definition of $C(u)$ implies that the operator commutator

$$[A(u), \partial/\partial p_1] = \alpha C(u). \quad (2.23)$$

Part (ii) of the corollary follows from our theorem by using the relation (2.23). By taking the relation adjoint to (2.23) relative to the bilinear form (2.7) we get

$$[p_1, A(u)] = \alpha B(u). \quad (2.24)$$

Part (i) of the corollary follows from our theorem by using the latter relation. \square

The relations (2.23) and (2.24) were used in [15, Sec.1] as definitions of the series $C(u)$ and $B(u)$ respectively. Our theorem provides explicit formulas for the series defined in that way. Hence we can explicitly construct the elementary shift operators for Jack symmetric functions, see the equalities (2.25) and (2.27).

Let the partition λ be fixed. Then for $i = 1, \dots, \ell(\lambda)$ the elements $\alpha \lambda_i - i + 1$ of the field $\mathbb{Q}(\alpha)$ are pairwise distinct. Therefore by the part (i) of the corollary for the partition μ corresponding to any of these indices i we have

$$B(\alpha \lambda_i - i + 1) P_\mu = B_{\lambda\mu}(\alpha \lambda_i - i + 1) P_\lambda \quad (2.25)$$

where the coefficient $B_{\lambda\mu}(\alpha\lambda_i - i + 1)$ is the product of (2.19) by

$$\prod_{j=1}^{\ell(\lambda)} \frac{1}{\alpha\lambda_i - i + j} \cdot \prod_{\substack{j=1 \\ j \neq i}}^{\ell(\lambda)} (\alpha\lambda_i - \alpha\lambda_j - i + j). \quad (2.26)$$

The left hand side of the equality (2.25) should be understood as the value in \mathcal{A} of the rational function $B(u)P_\mu$ at the point $u = \alpha\lambda_i - i + 1$. Similarly, by (ii)

$$C(\alpha\lambda_i - i + 1)P_\lambda = C_{\mu\lambda}(\alpha\lambda_i - i + 1)P_\mu \quad (2.27)$$

where the coefficient $C_{\mu\lambda}(\alpha\lambda_i - i + 1)$ is the product of (2.20) by (2.26).

2.7. Reduction of the proof. In this subsection we reduce the proof of our theorem to proving a certain determinantal identity for each $N = 1, 2, \dots$. This identity will be proved in the next section by using the induction on N .

By the lemma from Subsection 2.4 our theorem is equivalent to the equality

$$A(u)(\Pi)/\Pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{(u)_k} \sum_{\ell(\lambda)=k} d_\lambda m_\lambda(x_1, x_2, \dots) m_\lambda(y_1, y_2, \dots) \quad (2.28)$$

where the coefficients of the series $A(u)$ are regarded as operators acting on the symmetric functions in the variables x_1, x_2, \dots . Here we set $(u)_0 = 1$. It suffices to prove for each $N = 1, 2, \dots$ the restriction of the functional equality (2.28) to

$$x_{N+1} = x_{N+2} = \dots = 0. \quad (2.29)$$

By the very definition of $A(u)$ the restriction of the left hand side of (2.28) to (2.29) as of a function in the variables x_1, x_2, \dots equals

$$A_N(u)(\Pi_N)/\Pi_N \quad (2.30)$$

where we denote

$$\Pi_N = \prod_{i=1}^N \prod_{j=1}^{\infty} (1 - x_i y_j)^{-1/\alpha}.$$

Simply by the definition of the monomial symmetric function $m_\lambda(x_1, x_2, \dots)$ its restriction to (2.29) is $m_\lambda(x_1, \dots, x_N)$ if $\ell(\lambda) \leq N$ and vanishes if $\ell(\lambda) > N$. Therefore the restriction of the right hand side of (2.28) to (2.29) equals

$$\sum_{k=0}^N \frac{(-1)^k}{(u)_k} \sum_{\ell(\lambda)=k} d_\lambda m_\lambda(x_1, \dots, x_N) m_\lambda(y_1, y_2, \dots). \quad (2.31)$$

Further, due to [VI.2.19] to prove the equality of (2.30) to (2.31) it suffices to set

$$y_{N+1} = y_{N+2} = \dots = 0.$$

However, we will keep working with the infinite collection of variables y_1, y_2, \dots . This will simplify the induction argument in the next section.

Let us compute the function (2.30). It depends on the variable u rationally. It is also symmetric in either of the two collections of variables x_1, \dots, x_N and y_1, y_2, \dots . This function can be obtained by applying to the identity function 1 the result of conjugating $A_N(u)$ by the operator of multiplication by Π_N .

Conjugating the operator (2.9) by Π_N amounts to replacing each ∂_i in (2.9) with the sum

$$\partial_i + \alpha^{-1} \sum_{l=1}^{\infty} \frac{y_l}{1 - x_i y_l}.$$

Here we are just adding to each ∂_i the logarithmic derivative of the function Π_N relative to x_i . Hence conjugating (2.9) by the multiplication by Π_N yields

$$\sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma \prod_{i=1}^N \left(x_i^{N-\sigma(i)} \left(u + \sigma(i) - 1 - \alpha x_i \partial_i - \sum_{l=1}^{\infty} \frac{x_i y_l}{1 - x_i y_l} \right) \right).$$

Here in any single summand each of the factors corresponding to $i = 1, \dots, N$ does not depend on the variables x_j with $j \neq i$. Therefore when applying the latter operator sum to the identity function 1 we can simply replace each ∂_i with zero. Then we get the function

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma \prod_{i=1}^N \left(x_i^{N-\sigma(i)} \left(u + \sigma(i) - 1 - \sum_{l=1}^{\infty} \frac{x_i y_l}{1 - x_i y_l} \right) \right) = \\ \det \left[x_i^{N-j} \left(u + j - 1 + \sum_{l=1}^{\infty} \frac{x_i y_l}{x_i y_l - 1} \right) \right]_{i,j=1}^N. \end{aligned} \quad (2.32)$$

It follows that the function (2.30) is equal to the determinant (2.32) divided by the Vandermonde polynomial $\Delta(x_1, \dots, x_N)$ and by the Pochhammer symbol $(u)_N$, see (2.8) and (2.15). This ratio is equal to the right hand side of (2.31) by the proposition in Subsection 3.1, see the argument at the end of that subsection. We will prove the proposition in Subsections 3.2 to 3.4. Thus we will complete the proof of our theorem. Note that another proof of this theorem can be obtained by using the results of [2, Sec. 3] and [17, Sec. 9] on the Macdonald operators.

3. Determinantal identities

3.1. Getting the theorem. Let \mathbb{F} be any field. Consider the rational function of two variables u, v

$$\Psi(u, v) = \frac{u v}{u v - 1} \quad (3.1)$$

with values in \mathbb{F} . This function is a solution of the equation

$$(u - v) \Psi(u, w) \Psi(v, w) = u \Psi(v, w) - v \Psi(u, w). \quad (3.2)$$

Here w is a third variable. Any non-zero rational solution $\Psi(u, v)$ of (3.2) is

$$\frac{u}{u - \psi(v)}$$

where $\psi(v)$ is an arbitrary rational function of a single variable. In particular, here by choosing $\psi(v) = v^{-1}$ we get the solution (3.1).

Let x_1, \dots, x_N and y_1, y_2, \dots be independent variables. Here we assume that $N \geq 1$. In the next three subsections we will prove the following proposition.

Proposition. *For any solution $\Psi(u, v)$ of the equation (3.2) we have an identity*

$$\det \left[x_i^{N-j} \left(u + j - 1 + \sum_{l=1}^{\infty} \Psi(x_i, y_l) \right) \right]_{i,j=1}^N = \Delta(x_1, \dots, x_N) \sum_{k=0}^N (u+k) \dots (u+N-1) \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) \quad (3.3)$$

where all the indices $i_1, \dots, i_k \in \{1, \dots, N\}$ are distinct, the indices $j_1, \dots, j_k \in \{1, 2, \dots\}$ are all distinct too, and the sum is taken over all collections of these indices such that different are all the corresponding sets of k pairs

$$\{(i_1, j_1), \dots, (i_k, j_k)\}. \quad (3.4)$$

For any $k \geq 1$ take the symmetric group \mathfrak{S}_k . Using the permutations $\sigma \in \mathfrak{S}_k$ the sum over the indices i_1, \dots, i_k and j_1, \dots, j_k at the right hand side of the equality (3.3) can be also written as

$$\sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_k}} \sum_{\sigma \in \mathfrak{S}_k} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_{\sigma(r)}}). \quad (3.5)$$

Hence by choosing the function $\Psi(u, v)$ as in (3.1) our proposition implies that the determinant (2.32) equals

$$\Delta(x_1, \dots, x_N) \sum_{k=0}^N (u+k) \dots (u+N-1) \times \sum_{\ell(\lambda)=k} d_{\lambda} m_{\lambda}(x_1, \dots, x_N) m_{\lambda}(y_1, y_2, \dots). \quad (3.6)$$

Indeed, if $\Psi(u, v)$ is the rational function (3.1) then the sum (3.5) equals

$$\begin{aligned} & \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_k}} \sum_{\sigma \in \mathfrak{S}_k} \prod_{r=1}^k \frac{x_{i_r} y_{j_{\sigma(r)}}}{x_{i_r} y_{j_{\sigma(r)}} - 1} = \\ & \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_k}} \sum_{\sigma \in \mathfrak{S}_k} \sum_{l_1, \dots, l_k=1}^{\infty} \prod_{r=1}^k (x_{i_r} y_{j_{\sigma(r)}})^{l_r} = \\ & \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_k}} \sum_{\sigma, \tau \in \mathfrak{S}_k} \sum_{\ell(\lambda)=k} d_{\lambda}^{-1} \prod_{r=1}^k (x_{i_r} y_{j_{\sigma(r)}})^{\lambda_{\tau(r)}}. \end{aligned}$$

Here λ ranges over all partitions λ of length k . The sum displayed in the last line equals the sum in the second line of (3.6), by using the expression (2.1) for the polynomial $m_\lambda(x_1, \dots, x_N)$ and a similar expression for $m_\lambda(y_1, y_2, \dots)$. Thus our proposition implies the theorem as stated in Subsection 2.5.

3.2. Expanding the determinant. We will prove the proposition by induction on N . The case $N = 1$ is the induction base. Here the left hand side of (3.3) is

$$u + \sum_{l=1}^{\infty} \Psi(x_1, y_l).$$

The right hand side of (3.3) is then the same by definition, since $\Delta(x_1) = 1$.

Now take $N > 1$ and assume that the identity (3.3) holds for $N - 1$ instead of N . For each index $i = 1, \dots, N$ we will for short denote

$$\Delta_i = \Delta(x_1, \dots, \widehat{x}_i, \dots, x_N)$$

where as usual the symbol \widehat{x}_i indicates the omitted variable. By expanding the determinant at the left hand side of (3.3) in the first column and then using the induction assumption with $u + 1$ instead of u , we get the sum

$$\begin{aligned} & \sum_{i=1}^N (-1)^{i+1} x_i^{N-1} u \Delta_i \times \\ & \sum_{k=0}^{N-1} (u + k + 1) \dots (u + N - 1) \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) + \\ & \sum_{i=1}^N (-1)^{i+1} x_i^{N-1} \sum_{l=1}^{\infty} \Psi(x_i, y_l) \Delta_i \times \\ & \sum_{k=0}^{N-1} (u + k + 1) \dots (u + N - 1) \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}). \end{aligned} \quad (3.7)$$

Now consider the sum

$$\sum_{i=1}^N (-1)^{i+1} x_i^{N-1} \Delta_i \sum_{l=1}^{\infty} \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \Psi(x_i, y_l) \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r})$$

coming from the last two lines of the display (3.7). This sum can be written as

$$\begin{aligned} & \sum_{i=1}^N (-1)^{i+1} x_i^{N-1} \Delta_i \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \sum_{l \neq j_1, \dots, j_k} \Psi(x_i, y_l) \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) + \\ & \sum_{i=1}^N (-1)^{i+1} x_i^{N-1} \Delta_i \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \sum_{s=1}^k \Psi(x_i, y_{j_s}) \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}). \end{aligned} \quad (3.8)$$

We will prove that the sum in the second line of (3.8) equals

$$k \sum_{i=1}^N (-1)^{i+1} x_i^{N-1} \Delta_i \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}). \quad (3.9)$$

Due to that equality the sum (3.7) can be rewritten as

$$\begin{aligned} & \sum_{i=1}^N (-1)^{i+1} x_i^{N-1} \Delta_i \times \\ & \left(\sum_{k=0}^{N-1} (u+k)(u+k+1) \dots (u+N-1) \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) + \right. \\ & \left. \sum_{k=0}^{N-1} (u+k+1) \dots (u+N-1) \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k \neq l}} \Psi(x_i, y_l) \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) \right). \quad (3.10) \end{aligned}$$

In the second line of (3.10) we can include the index $k = N$ to the summation range without affecting the sum since k distinct indices $i_1, \dots, i_k \neq i$ exist only if $k < N$. In the third line we can replace $k+1$ with k where $k = 1, \dots, N$. We get

$$\sum_{k=1}^N (u+k) \dots (u+N-1) \sum_{\substack{i_1, \dots, i_{k-1}, i \\ j_1, \dots, j_{k-1}, l}} \Psi(x_i, y_l) \prod_{r=1}^{k-1} \Psi(x_{i_r}, y_{j_r}).$$

Here the set

$$\{((i_1, j_1), \dots, (i_{k-1}, j_{k-1}), (i, l))\}$$

can be any of the sets (3.4) appearing in (3.3), provided that in (3.4) one of the indices i_1, \dots, i_k coincides with the given index i . That one index can be then denoted by i_k because the order of the k elements of the set (3.4) does not matter.

These observations will show that the sum displayed in the second and the third lines of (3.10) equals

$$\sum_{k=0}^N (u+k) \dots (u+N-1) \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}).$$

In particular, they will show that this sum does not depend on the index i . Hence we will have the identity (3.3) proved, by expanding the determinant $\Delta(x_1, \dots, x_N)$ in the first column.

3.3. Two sums. We need to prove for $k = 0, \dots, N-1$ that the sum (3.9) equals the sum in the second line of (3.8). By using (3.2) the last mentioned sum can be written as

$$\sum_{i=1}^N \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \sum_{s=1}^k (-1)^{i+1} x_i^{N-1} \Delta_i \times \\ \frac{x_i \Psi(x_{i_s}, y_{j_s}) - x_{i_s} \Psi(x_i, y_{j_s})}{x_i - x_{i_s}} \prod_{r \neq s} \Psi(x_{i_r}, y_{j_r})$$

which in turn can be rewritten as

$$\begin{aligned} & \sum_{i=1}^N \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \sum_{s=1}^k (-1)^{i+1} x_i^{N-1} \Delta_i \prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) + \\ & \sum_{i=1}^N \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \sum_{s=1}^k (-1)^{i+1} x_i^{N-1} \Delta_i \times \\ & \left(\frac{x_{i_s} \Psi(x_{i_s}, y_{j_s})}{x_i - x_{i_s}} + \frac{x_{i_s} \Psi(x_i, y_{j_s})}{x_{i_s} - x_i} \right) \prod_{r \neq s} \Psi(x_{i_r}, y_{j_r}). \end{aligned} \quad (3.11)$$

The summands in the first line of the display (3.11) do not depend on s . Hence their sum equals (3.9). Let us show that the sum appearing in the second and the third lines of (3.11) equals zero. By opening the brackets in the third line and then swapping the running indices i, i_s in each term coming from the second fraction in the brackets, the sum in the second and the third lines becomes

$$\begin{aligned} & \sum_{i=1}^N \sum_{\substack{i_1, \dots, i_k \neq i \\ j_1, \dots, j_k}} \sum_{s=1}^k \left(\prod_{r=1}^k \Psi(x_{i_r}, y_{j_r}) \right) \times \\ & \frac{(-1)^{i+1} x_i^{N-1} x_{i_s} \Delta_i + (-1)^{i_s+1} x_{i_s}^{N-1} x_i \Delta_{i_s}}{x_i - x_{i_s}} \end{aligned}$$

Here in the first line the product over $r = 1, \dots, k$ depends neither on the index s nor on the choice of the index $i \neq i_1, \dots, i_k$. The fraction in the second line does not depend on the indices j_1, \dots, j_k . We will show that for any *fixed* distinct indices $i_1, \dots, i_k \in \{1, \dots, N\}$ the sum of these fractions over $s = 1, \dots, k$ and $i \neq i_1, \dots, i_k$ is equal to zero. This will complete our proof of the identity (3.3).

Let σ range over \mathfrak{S}_N . By the definition of the Vandermonde polynomial, the sum of the last displayed fractions over $s = 1, \dots, k$ and $i \neq i_1, \dots, i_k$ equals

$$\sum_{i \neq i_1, \dots, i_k} \sum_{s=1}^k \sum_{\sigma(i)=1} (-1)^\sigma \frac{x_i^{N-1} x_{i_s}^{N-\sigma(i_s)+1} - x_{i_s}^{N-1} x_i^{N-\sigma(i_s)+1}}{x_i - x_{i_s}} \prod_{j \neq i, i_s} x_j^{N-\sigma(j)}$$

which is in turn equal to the sum

$$\sum_{i \neq i_1, \dots, i_k} \sum_{s=1}^k \sum_{\sigma(i)=1} (-1)^\sigma \sum_{r=2}^{\sigma(i_s)-1} x_i^{N-r} x_{i_s}^{N-\sigma(i_s)+r-1} \prod_{j \neq i, i_s} x_j^{N-\sigma(j)}. \quad (3.12)$$

Recall that here $N > 1$. One can easily prove by induction on $N = 2, 3, \dots$ that the total number of terms in the sum (3.12) equals

$$(N-1)!(N-2)(N-k)k/2.$$

However, we shall not use this equality and will leave its proof to the reader. In the next subsection, we will prove that all terms in (3.12) cancel each other.

3.4. Cancellations. The sum (3.12) is taken over quadruples (i, s, σ, r) . Take any of them such that

$$\sigma^{-1}(r) \neq i_1, \dots, i_k.$$

Such a quadruple will be called *of type I*. Denote $\sigma^{-1}(r) = \bar{i}$. Observe that $\bar{i} \neq i$ because $\sigma(\bar{i}) = r \geq 2$ while $\sigma(i) = 1$. Put

$$\bar{\sigma} = \sigma \tau_{i\bar{i}} \quad (3.13)$$

where $\tau_{i\bar{i}} \in \mathfrak{S}_N$ is the transposition of i and \bar{i} . Then $\bar{\sigma}(i) = r$ and $\bar{\sigma}(\bar{i}) = 1$. We also have $\bar{\sigma}(i_s) = \sigma(i_s)$ because $i_s \neq i, \bar{i}$. Hence the quadruple $(\bar{i}, s, \bar{\sigma}, r)$ appears in (3.12) together with the quadruple (i, s, σ, r) . But the summands in (3.12) corresponding to these two quadruples cancel each other. Indeed,

$$\begin{aligned} & -(-1)^\sigma x_i^{N-r} x_{\bar{i}}^{N-\sigma(\bar{i})} x_{i_s}^{N-\sigma(i_s)+r-1} \prod_{j \neq i, \bar{i}, i_s} x_j^{N-\sigma(j)} = \\ & (-1)^{\bar{\sigma}} x_i^{N-\bar{\sigma}(i)} x_{\bar{i}}^{N-r} x_{i_s}^{N-\bar{\sigma}(i_s)+r-1} \prod_{j \neq i, \bar{i}, i_s} x_j^{N-\bar{\sigma}(j)}. \end{aligned}$$

Note that here

$$\bar{\sigma}^{-1}(r) = i \neq i_1, \dots, i_k$$

hence $(\bar{i}, s, \bar{\sigma}, r)$ is also of type I. Moreover, by applying our construction to the latter quadruple instead of (i, s, σ, r) we get the initial quadruple (i, s, σ, r) back.

Next take a quadruple (i, s, σ, r) showing in (3.12) such that for some index \bar{s}

$$\sigma(i_s) - r + 1 = \sigma(i_{\bar{s}}).$$

Such a quadruple will be called *of type II*. Note that $s \neq \bar{s}$ because $r \neq 1$. Put

$$\bar{\sigma} = \sigma \tau_{i_s i_{\bar{s}}}.$$

Here $\bar{\sigma}(i) = \sigma(i) = 1$ because $i \neq i_s, i_{\bar{s}}$. We also have $\bar{\sigma}(i_{\bar{s}}) = \sigma(i_s)$. Hence the quadruple $(i, \bar{s}, \bar{\sigma}, r)$ appears in (3.12) together with (i, s, σ, r) . The summands in (3.12) corresponding to these two quadruples cancel each other. Indeed,

$$\begin{aligned}
& -(-1)^\sigma x_i^{N-r} x_{i_s}^{N-\sigma(i_s)+r-1} x_{i_{\bar{s}}}^{N-\sigma(i_{\bar{s}})} \prod_{j \neq i, i_s, i_{\bar{s}}} x_j^{N-\sigma(j)} = \\
& (-1)^{\bar{\sigma}} x_i^{N-r} x_{i_s}^{N-\bar{\sigma}(i_s)} x_{i_{\bar{s}}}^{N-\bar{\sigma}(i_{\bar{s}})+r-1} \prod_{j \neq i, i_s, i_{\bar{s}}} x_j^{N-\bar{\sigma}(j)}.
\end{aligned}$$

Here

$$\bar{\sigma}(i_{\bar{s}}) - r + 1 = \sigma(i_s) - r + 1 = \bar{\sigma}(i_s)$$

and $(i, \bar{s}, \bar{\sigma}, r)$ is also of type II. Moreover, by applying our construction to the latter quadruple instead of (i, s, σ, r) we get the initial quadruple (i, s, σ, r) back.

Note that the above two constructions differ for the quadruples of type I and II, while a quadruple can be of both types simultaneously. However, the summands in (3.12) corresponding to quadruples of any of the two types still cancel each other. Indeed, take any quadruple (i, s, σ, r) of type I which also has type II. By applying our first construction to it we get another quadruple $(\bar{i}, s, \bar{\sigma}, r)$ of type I, where $\bar{\sigma}$ is defined by (3.13) while $\bar{i} = \sigma^{-1}(r)$. But then for a certain index \bar{s} we have

$$\bar{\sigma}(i_s) - r + 1 = \sigma(i_s) - r + 1 = \sigma(i_{\bar{s}})$$

because (i, s, σ, r) also has type II. Hence the quadruple $(\bar{i}, s, \bar{\sigma}, r)$ is of type II too. We could similarly check that the result of applying our second construction to (i, s, σ, r) is not only of type II but of type I as well. However, this is already not needed for the cancellation. So we will leave checking it to the reader.

Finally, take any quadruple (i, s, σ, r) appearing in (3.12) which is neither of type I nor of type II. Such a quadruple will be called *of type III*. Here $r = \sigma(i_{\bar{s}})$ for some index \bar{s} because (i, s, σ, r) is not of type I. For some index $\bar{i} \neq i_1, \dots, i_k$ we also have

$$\sigma(i_s) - r + 1 = \sigma(\bar{i}) \tag{3.14}$$

because (i, s, σ, r) is not of type II. Observe that here the four indices $i, i_{\bar{s}}, i_s, \bar{i}$ are pairwise distinct. Indeed, here we have $i, \bar{i} \neq i_s, i_{\bar{s}}$ by definition. Further, here $i \neq \bar{i}$ because $\sigma(i) = 1$ while $\sigma(\bar{i}) \geq 2$ by the definition (3.14). Furthermore, here $i_s \neq i_{\bar{s}}$ because $r < \sigma(i_s)$ while $r = \sigma(i_{\bar{s}})$. Put $\bar{\sigma} = \sigma\tau$ where $\tau \in \mathfrak{S}_N$ cyclically permutes the indices $i, i_{\bar{s}}, i_s, \bar{i}$ and leaves all the remaining indices fixed. More exactly,

$$\tau : i \mapsto i_{\bar{s}} \mapsto i_s \mapsto \bar{i} \mapsto i.$$

Let \bar{r} be the number at either side of equality (3.14). Consider the quadruple $(\bar{i}, \bar{s}, \bar{\sigma}, \bar{r})$. Here $\bar{\sigma}(\bar{i}) = \sigma(i) = 1$ by definition. Due to the range of r we also have

$$2 \leq \bar{r} \leq \sigma(i_s) - 1 = \bar{\sigma}(i_{\bar{s}}) - 1.$$

Therefore the quadruple $(\bar{i}, \bar{s}, \bar{\sigma}, \bar{r})$ appears in (3.12) together with (i, s, σ, r) . The summands in (3.12) corresponding to the two quadruples cancel each other. Indeed, we have the equality

$$\begin{aligned}
 & -(-1)^\sigma x_i^{N-r} x_{i_{\bar{s}}}^{N-\sigma(i_{\bar{s}})} x_{i_s}^{N-\sigma(i_s)+r-1} x_{\bar{i}}^{N-\sigma(\bar{i})} \prod_{j \neq i, i_{\bar{s}}, i_s, \bar{i}} x_j^{N-\sigma(j)} = \\
 & (-1)^{\bar{\sigma}} x_i^{N-\bar{\sigma}(i)} x_{i_{\bar{s}}}^{N-\bar{\sigma}(i_{\bar{s}})+\bar{r}-1} x_{i_s}^{N-\bar{\sigma}(i_s)} x_{\bar{i}}^{N-\bar{\sigma}} \prod_{j \neq i, i_{\bar{s}}, i_s, \bar{i}} x_j^{N-\bar{\sigma}(j)}.
 \end{aligned}$$

Here $\bar{r} = \bar{\sigma}(i_s)$ so that $(\bar{i}, \bar{s}, \bar{\sigma}, \bar{r})$ is not of type I. We also have

$$\bar{\sigma}(i_{\bar{s}}) - \bar{r} + 1 = \bar{\sigma}(i)$$

so that $(\bar{i}, \bar{s}, \bar{\sigma}, \bar{r})$ is not of type II. So this quadruple is of type III. Moreover, by applying our third construction to this quadruple instead of (i, s, σ, r) we get the initial quadruple (i, s, σ, r) back. Thus all summands in (3.12) cancel each other. We have now completed the induction step in the proof of our proposition.

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References

- .1 A. G. Abanov and P. B. Wiegmann, *Quantum Hydrodynamics, the quantum Benjamin-Ono equation, and the Calogero model*, Phys. Rev. Lett. **95** (2005), 076402.
- .2 H. Awata and H. Kanno, *Macdonald operators and homological invariants of the colored Hopf link*, J. Phys. **A 44** (2011), 375201.
- .3 H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, *Collective fields, Calogero-Sutherland model and generalized matrix models*, Phys. Lett. **B 347** (1995), 49–55.
- .4 W. Cai and N. Jing, *Applications of Laplace-Beltrami operator for Jack polynomials*, European J. Combin. **33** (2012), 556–571.
- .5 F. Calogero, *Ground state of a one-dimensional N-body system*, J. Math. Phys. **10** (1969), 2197–2200.
- .6 A. Debiard, *Polynômes de Tchêbychev et de Jacobi dans un espace euclidien de dimension p*, C. R. Acad. Sc. Paris **I 296** (1983), 529–532.
- .7 S. Iso, *Anyon basis of c = 1 Conformal Field Theory*, Nucl. Phys. **B 443** (1995), 581–595.
- .8 I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.
- .9 J. A. Minahan and A. P. Polychronakos, *Density correlation functions in Calogero-Sutherland models*, Phys. Rev. **B 50** (1994), 4236–4239.
- .10 M. L. Nazarov and E. K. Sklyanin, *Macdonald operators at infinity*, submitted.
- .11 A. Okounkov and R. Pandharipande, *Quantum cohomology of the Hilbert scheme of points in the plane*, Inv. Math. **179** (2010), 523–557.
- .12 A. K. Pogrebkov, *Boson-fermion correspondence and quantum integrable and dispersionless models*, Russ. Math. Surv. **58** (2003), 1003–1037.
- .13 A. P. Polychronakos, *Waves and solitons in the continuum limit of the Calogero-Sutherland model*, Phys. Rev. Lett. **74** (1995), 5153–5157.
- .14 P. Rossi, *Gromov-Witten invariants of target curves via Symplectic Field Theory*, J. Geom. Phys. **58** (2008), 931–941.
- .15 O. Schiffmann and E. Vasserot, *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on \mathbb{A}^2* , arXiv:1202.2756

- .16 J. Sekiguchi, *Zonal spherical functions on some symmetric spaces*, Publ. Res. Inst. Math. Sci. **12** (1977), 455–459.
- .17 J. Shiraishi, *A family of integral transformations and basic hypergeometric series*, Comm. Math. Phys. **263** (2006), 439–460.
- .18 E. K. Sklyanin, *Separation of variables. New trends*, Progress Theor. Phys. Suppl. **118** (1995), 35–60.
- .19 B. Sutherland, *Exact results for a quantum many-body problem in one dimension*, Phys. Rev. **A 4** (1971), 2019–2021.
- .20 B. Sutherland, *Exact results for a quantum many-body problem in one dimension II*, Phys. Rev. **A 5** (1972), 1372–1376.